

The Haag-Kastler Axioms for the $\mathcal{P}(\varphi)_2$ Model on the De Sitter Space

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Abstract

We establish the Haag-Kastler axioms for a class of models on the two-dimensional de Sitter space, including the $\mathcal{P}(\varphi)_2$ model.

1 Introduction

In recent work [2], we have provided a novel construction of the $\mathcal{P}(\varphi)_2$ model on the de Sitter space. Our work was presented in the canonical formalism, and although finite speed of light was established, we find it worthwhile to provide a proof of the Haag-Kastler axioms.

2 One-particle space

The two-dimensional de Sitter space

$$\mathbb{dS} \doteq \{x \in \mathbb{R}^{1+2} \mid x_0^2 - x_1^2 - x_2^2 = -r^2\} , \quad r > 0 , \quad (1)$$

can be viewed as a one-sheeted *hyperboloid*, embedded in the $(1+2)$ -dimensional Minkowski space \mathbb{R}^{1+2} . The embedding (1) is compatible with the metric and the causal structure, *i.e.*, the de Sitter space \mathbb{dS} inherits its metric and the causal structure from the ambient Minkowski space.

The isometry group of \mathbb{dS} is $O(1, 2)$. The connected component containing the identity is $SO_0(1, 2)$. The latter is generated by the *rotations*

$$R_0(\alpha) \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} , \quad \alpha \in [0, 2\pi) ,$$

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and the *Lorentz boosts*

$$\Lambda_1(t) \doteq \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}, \quad t \in \mathbb{R}.$$

According to our convention, the boosts $\Lambda_1(t)$ keep the x_1 -axis invariant, and therefore correspond to boosts in the x_2 -direction.

The circle

$$S^1 \doteq \{x \in \mathbb{dS} \mid x_0 = 0\}$$

forms a *Cauchy surface* for \mathbb{dS} . For two points $x = (0, r \sin \psi, r \cos \psi)$ and $y = (0, r \sin \psi', r \cos \psi')$ on the circle S^1 , the Wightman two-point function of a scalar free field (analysed in [4]) equals

$$\mathcal{W}^{(2)}(x, y) = c_v P_{s^+}(-\cos(\psi - \psi')). \quad (2)$$

Here P_{s^+} is the *Legendre function* for the parameter

$$s^\pm = -\frac{1}{2} \mp i\nu, \quad (3)$$

with

$$\nu = \begin{cases} i\sqrt{\frac{1}{4} - \zeta^2} & \text{if } 0 < \zeta < 1/2, \\ \sqrt{\zeta^2 - \frac{1}{4}} & \text{if } \zeta \geq 1/2, \end{cases}$$

and ζ the eigenvalue of the Casimir operator of $SO_0(1, 2)$.

The two-point function (2) gives rise to a scalar product on $C^\infty(S^1)$:

$$\begin{aligned} \langle h, h' \rangle_{\mathcal{H}} &\doteq c_v \int_{S^1} r d\psi \int_{S^1} r d\psi' \overline{h(\psi)} \\ &\quad \times P_{s^+}(-\cos(\psi - \psi')) h'(\psi'). \end{aligned} \quad (4)$$

The value of the positive normalisation constant is

$$c_v = -\frac{1}{2 \sin(\pi s^+)} = \frac{1}{2 \cos(i\nu\pi)}.$$

Note that the singularity for $\psi = \psi'$ is integrable. The completion of $C^\infty(S^1)$ w.r.t. this scalar product is a one-particle Hilbert space, which we denote by \mathcal{H} .

Lemma 2.1. *The scalar product (4) can be expressed as*

$$\langle h, h' \rangle_{\mathcal{H}} = \langle h, \frac{1}{2\omega} h' \rangle_{L^2(S^1, r d\psi)},$$

with ω a strictly positive self-adjoint operator on $L^2(S^1, r d\psi)$ with Fourier coefficients

$$\tilde{\omega}(k) = r^{-1} (k + s^+) \frac{\Gamma\left(\frac{k+s^+}{2}\right)}{\Gamma\left(\frac{k-s^+}{2}\right)} \frac{\Gamma\left(\frac{k+1-s^+}{2}\right)}{\Gamma\left(\frac{k+1+s^+}{2}\right)}, \quad k \in \mathbb{Z}.$$

One of the key results in [2] is that \mathcal{H} carries a *unitary irreducible representation* of the Lorentz group:

Theorem 2.2. *The rotations*

$$(\mathbf{u}(\mathbf{R}_0(\alpha))\mathbf{h})(\psi) = \mathbf{h}(\psi - \alpha) , \quad \alpha \in [0, 2\pi) , \mathbf{h} \in \mathcal{H} ,$$

and the boosts

$$\mathbf{u}(\Lambda_1(t)) = e^{it\omega r \cos\psi} , \quad t \in \mathbb{R} ,$$

generate a representation of $SO_0(1, 2)$ on \mathcal{H} .

Definition 2.3. Let I_+ be the open half-circle

$$I_+ \doteq \{x \in S^1 \mid x_2 > 0\} .$$

We now define \mathbb{R} -linear subspaces of \mathcal{H} :

i.) For the wedge

$$W_1 \doteq \{x \in \mathbb{d}S \mid x_2 > |x_0|\}$$

we set

$$\mathcal{H}(W_1) \doteq \{\mathbf{h} \in \mathcal{H} \mid \text{supp } (\Re \mathbf{h}, \omega^{-1} \Im \mathbf{h}) \subset I_+ \times I_+\} .$$

ii.) For an arbitrary wedge $W = \Lambda W_1$, $\Lambda \in SO_0(1, 2)$, we set

$$\mathcal{H}(W) \doteq \mathbf{u}(\Lambda)\mathcal{H}(W_1) . \tag{5}$$

iii.) For a causally complete, open and bounded region \mathcal{O} , we set

$$\mathcal{H}(\mathcal{O}) \doteq \bigcap_{\mathcal{O} \subset W} \mathcal{H}(W) . \tag{6}$$

The net $\mathcal{O} \mapsto \mathcal{H}(\mathcal{O})$ has a number of interesting properties, which will give rise to the local structure of free quantum fields on the de Sitter space.

Proposition 2.4. *The subspaces introduced in Definition 2.3 have the following properties:*

i.) (Wedge Duality). The \mathbb{R} -linear subspace $\mathcal{H}(W')$ for the opposite wedge

$$W' \doteq \{x \in \mathbb{d}S \mid x \text{ space-like separated from } W\}$$

equals the symplectic complement

$$\mathcal{H}(W)' \doteq \{\mathbf{h} \in \mathcal{H} \mid \Im \langle \mathbf{h}, \mathbf{g} \rangle = 0 \quad \forall \mathbf{g} \in \mathcal{H}(W)\}$$

of $\mathcal{H}(W)$.

ii.) (Localisation). For I a bounded open interval in S^1 , let $\mathcal{O}_I = I''$ denote the causal completion of the interval I in \mathcal{dS} . Then

$$\mathcal{H}(\mathcal{O}_I) = \{h \in \mathcal{H} \mid \text{supp } (\Re h, \omega^{-1} \Im h) \subset I \times I\}.$$

iii.) (Covariance). For \mathcal{O} a causally complete, open and bounded regions \mathcal{O} and $\Lambda \in SO_0(1,2)$, one finds

$$\mathcal{H}(\Lambda \mathcal{O}) = u(\Lambda) \mathcal{H}(\mathcal{O}).$$

In particular,

$$u(\Lambda_1(t)) \mathcal{H}(W_1) = \mathcal{H}(W_1) \quad \forall t \in \mathbb{R}.$$

iv.) (Microcausality). For two space-like separated causally complete, open and bounded regions \mathcal{O}_1 and \mathcal{O}_2 , one finds

$$\Im \langle h_1, h_2 \rangle_{\mathcal{H}} = 0 \quad \forall h_i \in \mathcal{H}(\mathcal{O}_i), \quad i = 1, 2.$$

Proof. For the special case $W = W_1$, property i.) follows from the definition of $\mathcal{H}(W_1)$, the definition of the symplectic complement, and the fact that

$$L^2(I, d\psi)^\perp = L^2(I^c, d\psi).$$

The general case follows from definition (5).

Next, let

$$W(\alpha) \doteq R_0(\alpha) W_1, \quad \alpha \in [0, 2\pi),$$

be a wedge whose edges lies on S^1 . As \mathcal{O}_I is causally complete,

$$\bigcap_{\mathcal{O}_I \subset W} W = \mathcal{O}_I = W(\alpha) \cap W(\beta)$$

for some fixed $\alpha, \beta \in [0, 2\pi)$. Inspecting the definitions, we find that

$$\{h \in \mathcal{H} \mid \text{supp } (\Re h, \omega^{-1} \Im h) \subset I \times I\}$$

is equal to

$$\mathcal{H}(W(\alpha)) \cap \mathcal{H}(W(\beta)).$$

As both $W(\alpha)$ and $W(\beta)$ are wedges which contain \mathcal{O}_I , we have

$$\mathcal{H}(\mathcal{O}_I) \subseteq \mathcal{H}(W(\alpha)) \cap \mathcal{H}(W(\beta)).$$

Next, we assume that W is an arbitrary wedge which contains \mathcal{O}_I . The opposite wedge W' of W is, like any wedge, of the form $\Lambda^{(\beta)}(t) R_0(\alpha) W_1$ for suitable α, β and t .

As a consequence of finite speed of light (see Proposition 4.10.2 in [2]), we have

$$\begin{aligned} \mathcal{H}(W') &= u(\Lambda^{(\beta)}(t)) \mathcal{H}(W(\alpha)) \\ &\subset \{h \in \mathcal{H} \mid \text{supp } (\Re h, \omega^{-1} \Im h) \subset J \times J\}, \end{aligned}$$

with $J = \Gamma(\Lambda^{(\beta)}(t) R_0(\alpha) I_+) \cap S^1$, where $\Gamma(M)$ is the domain of dependence of a set M , i.e., the union of the future $\Gamma^+(M)$ and the past $\Gamma^-(M)$ of M .

Note that

$$\text{i.) } \Gamma(\Lambda^{(\beta)}(\mathfrak{t})\mathbb{R}_0(\alpha)I_+) = \Gamma(W');$$

$$\text{ii.) } W' \text{ is space-like to } I, \text{ since } W \text{ contains } \mathcal{O}_I.$$

Hence $\Gamma(W') \cap S^1$ is in the interior $I^c \doteq S^1 \setminus \bar{I}$ of the complement of I within S^1 . Thus

$$\mathcal{H}(W') \subset \{h \in \mathcal{H} \mid \text{supp } (\Re h, \omega^{-1} \Im h) \subset I^c \times I^c\}.$$

It follows that

$$\begin{aligned} \mathcal{H}(W) &= \mathcal{H}(W')' \\ &\supseteq \underbrace{\{h \in \mathcal{H} \mid \text{supp } (\Re h, \omega^{-1} \Im h) \subset I \times I\}}_{=\mathcal{H}(W(\alpha)) \cap \mathcal{H}(W(\beta))}. \end{aligned}$$

This verifies property ii.). In order to verify property iii.), we compute

$$\begin{aligned} \mathcal{H}(\Lambda \mathcal{O}) &= \bigcap_{\Lambda \mathcal{O} \subset \Lambda W} \mathcal{H}(\Lambda W) = \bigcap_{\mathcal{O} \subset W} \mathcal{H}(\Lambda W) \\ &= \bigcap_{\mathcal{O} \subset W} u(\Lambda) \mathcal{H}(W) \\ &= u(\Lambda) \left(\underbrace{\bigcap_{\mathcal{O} \subset W} \mathcal{H}(W)}_{=\mathcal{H}(\mathcal{O})} \right). \end{aligned}$$

Finally, property iv.) follows from the fact that if \mathcal{O}_1 and \mathcal{O}_2 are two space-like separated causally complete, open and bounded regions, then there exists a wedge $W = \Lambda W_1$ such that

$$\mathcal{O}_1 \subset W \quad \text{and} \quad \mathcal{O}_2 \subset W'.$$

Applying $u(\Lambda)$ to the identity $\mathcal{H}(W_1)' = \mathcal{H}(W_1)'$, we see that locality is a consequence of covariance and wedge duality. \square

3 Second Quantization

The bosonic *Fock space* $\mathbb{F}(\mathcal{H})$ over \mathcal{H} is defined as the direct sum of the n -particle spaces:

$$\mathbb{F}(\mathcal{H}) \doteq \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s^n}, \quad \mathcal{H}^{\otimes_s^0} \doteq \mathbb{C},$$

with $\mathcal{H}^{\otimes_s^n}$ the n -fold totally symmetric tensor product \otimes_s of \mathcal{H} with itself. The *coherent vectors*

$$\mathbb{F}(h) \doteq \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \underbrace{h \otimes_s \cdots \otimes_s h}_{n\text{-times}}$$

form a total set in \mathcal{F} . The vector $\Omega_o = \mathbb{F}(0)$ is called the Fock vacuum. One can also define second quantized operators. Let A be a closed, densely defined linear operator on \mathcal{F} with domain $\mathcal{D}(A)$. Then

$$\mathbb{F}(A) : \mathcal{F} \rightarrow \mathcal{F}$$

is the closure of the linear operator acting on the linear combinations of coherent vectors with exponent in $\mathcal{D}(A)$ such that

$$\mathbb{F}(A)\mathbb{F}(h) = \mathbb{F}(Ah) .$$

This exponentiation preserves self-adjointness, positivity and unitarity.

4 Nets of Local Algebras

For $h, g \in \mathcal{H}$, the relations

$$\begin{aligned} V(h)V(g) &= e^{-i\mathcal{I}\langle h, g \rangle} V(h+g) , \\ V(h)\Omega_o &= e^{-\frac{1}{2}\|h\|^2} \mathbb{F}(ih) , \end{aligned}$$

define unitary operators, called the *Weyl operators*. They satisfy

$$V^*(h) = V(-h) \quad \text{and} \quad V(0) = 1 .$$

We use the Weyl operators to associate a von Neumann algebras acting on the Fock space \mathcal{F} to the wedge W_1 : let $\mathcal{A}_o(W_1)$ denote the von Neumann algebra generated by the Weyl operators

$$\{V(h) \mid h \in \mathcal{H}(W_1)\} .$$

Given a representation $U(\Lambda)$, $\Lambda \in SO_0(1, 2)$, of the Lorentz group acting on Fock space which acts kinetically on the Cauchy surface S^1 , we can than define von Neumann algebras associated to arbitrary bounded, causally complete, convex regions. We proceed in steps, repeating the ideas which lie behind Definition 2.3 and starting from the free algebra of the wedge W_1 . Note that $\Lambda W_1 \subset W_1 \Leftrightarrow \Lambda = \Lambda_1(t)$ for some $t \in \mathbb{R}$.

Definition 4.1. *Given a unitary representation $\Lambda \mapsto U(\Lambda)$ of the Lorentz group $SO_0(1, 2)$ acting on Fock space \mathcal{F} and satisfying the condition*

$$U(\Lambda_1(t))\mathcal{A}_o(W_1)U(\Lambda_1(t))^{-1} = \mathcal{A}_o(W_1), \quad t \in \mathbb{R},$$

we define the following von Neumann algebras:

i.) *For an arbitrary wedge $W = \Lambda W_1$, $\Lambda \in SO_0(1, d)$, we set*

$$\mathcal{A}(W) \doteq U(\Lambda)\mathcal{A}_o(W_1)U(\Lambda)^{-1} . \tag{7}$$

ii.) For an arbitrary bounded, causally complete, convex region (these are the de Sitter analogs of the double cones) $\mathcal{O} \subset \mathbb{dS}$, we set

$$\mathcal{A}(\mathcal{O}) \doteq \bigcap_{W \supset \mathcal{O}} \mathcal{A}(W) . \quad (8)$$

The inclusion preserving map

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$$

is called the net of local von Neumann algebras for the bosonic field on the de Sitter space \mathbb{dS} transforming under \mathbb{U} .

Remarks 4.1.

i.) In case $\mathbb{U} \equiv \mathbb{U}_\circ$,

$$\mathbb{U}_\circ(\Lambda) \doteq \mathbb{F}(\mathbf{u}(\Lambda)) , \quad \Lambda \in \mathrm{SO}_0(1,2) ,$$

we will denote the generator of one-parameter unitary group $\mathbf{t} \mapsto \mathbb{U}_\circ(\Lambda_1(\mathbf{t}))$ by L_\circ and the local algebra by $\mathcal{A}_\circ(\mathcal{O})$. It follows from a result by Araki (Theorem 1 in [1]) and Proposition 2.4 ii.) that

$$\mathcal{A}_\circ(\mathcal{O}_I) = \{V(\mathbf{h}) \mid \mathbf{h} \in \mathcal{H}, \text{supp } (\Re \mathbf{h}, \omega^{-1} \Im \mathbf{h}) \subset I \times I\}'' ;$$

just as one might have expected.

ii.) For the $\mathcal{P}(\varphi)_2$ model on the de Sitter space, the representation \mathbb{U} is given as follows.

a.) The rotations are the free ones,

$$\mathbb{U}(\mathbf{R}_0(\alpha)) = \mathbb{F}(\mathbf{u}(\mathbf{R}_0(\alpha))), \quad \alpha \in [0, 2\pi); \quad (9)$$

b.) The generator of the one-parameter unitary group $\mathbf{t} \mapsto \mathbb{U}(\Lambda_1(\mathbf{t}))$ can be expressed in terms of canonical fields and canonical momenta (see [2] for details):

$$L = L_\circ + \lim_{\epsilon \rightarrow 0} \int_{S^1} r \cos \psi \, d\psi : \mathcal{P}(\varphi(\delta_\epsilon(\cdot - \psi))) : ,$$

where \mathcal{P} is a polynomial, bounded from below, $\varphi(\mathbf{h})$ is the generator of the one-parameter unitary group $s \mapsto V(s\mathbf{h})$, and δ_ϵ approximates the Dirac delta function as $\epsilon \rightarrow 0$. As usual, the $::$ indicates normal ordering.

In the general case, we will need a criterium for the representation of the Lorentz group which ensures that the intersection in (8) is not trivial. A sufficient condition is the following: assume the von Neumann algebras are defined as in Definition 4.1. The net of local algebras is said to satisfy *finite speed of light*,

if for any wedge W , the algebra $\mathcal{A}(W)$ is contained in the time-zero Weyl algebra

$$\{V(h) \mid h \in \mathcal{H}; \text{supp } (\Re h, \omega^{-1}\Im h) \subset J \times J\}'' , \quad (10)$$

where $J = \Gamma(W) \cap S^1$.

Theorem 4.2. *Assume the net of local algebras satisfies finite speed of light. Then the local algebras associated to an interval $I \subset S^1$ on the Cauchy surface coincide with those of the free theory, i.e.,*

$$\mathcal{A}(\mathcal{O}_I) = \mathcal{A}_o(\mathcal{O}_I) , \quad I \subset S^1 .$$

Proof. The proof follows the ideas exposed in the proof of Proposition 2.4 ii.). Thus the key step is to show that for any wedge W which contains \mathcal{O}_I , we have

$$\mathcal{A}(W') \subset \{V(h) \mid h \in \mathcal{H}, \text{supp } (\Re h, \omega^{-1}\Im h) \subset I^c \times I^c\}'' ,$$

where $I^c \doteq S^1 \setminus \bar{I}$. As the edges of W are necessarily space- or light-like to \mathcal{O}_I , this inclusion follows from finite speed of light as expressed in (10). By duality,

$$\mathcal{A}(W) \supset \underbrace{\{V(h) \mid h \in \mathcal{H}, \text{supp } (\Re h, \omega^{-1}\Im h) \subset I \times I\}}_{=\mathcal{A}_o(\mathcal{O}_I)}'' ,$$

whenever W includes \mathcal{O}_I . □

Remarks 4.2.

- i.) For the $\mathcal{P}(\varphi)_2$ model on the de Sitter space, property (10), which encodes finite speed of light, was established in Theorem 10.1.1 in [2].
- ii.) The circle S^1 , which we use to identify the free field and the interacting field, could be replaced by any space-like geodesic ΛS^1 , $\Lambda \in \text{SO}_0(1, 2)$.

5 The Haag-Kastler Axioms

Assume we have defined a net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ of local algebras according to Definition 4.1, with the rotations being implemented by the free representation (9). In case this net respects *finite speed of light*, it will also satisfy the following Haag-Kastler axioms:

Theorem 5.1. *The net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ shares the following properties:*

- i.) (Locality). *The local algebras satisfy*

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)' \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2' .$$

Here \mathcal{O}' denotes the space-like complement of \mathcal{O} in \mathbb{dS} and $\mathcal{A}(\mathcal{O})'$ is the commutant of $\mathcal{A}(\mathcal{O})$ in $\mathcal{B}(\mathcal{F})$.

- ii.) (Covariance). The representation $\mathbb{U}: \Lambda \mapsto \mathbb{U}(\Lambda)$ act geometrically, i.e., for $\Lambda \in \text{SO}_0(1, 2)$,

$$\mathbb{U}(\Lambda)\mathcal{A}(\mathcal{O})\mathbb{U}(\Lambda)^{-1} = \mathcal{A}(\Lambda\mathcal{O}) .$$

- iii.) (Existence and Uniqueness of the Vacuum[3]). There exists a unique (up to a phase) unit vector $\Omega \in \mathcal{F}$, which

- a.) is invariant under the action of $\mathbb{U}(\text{SO}_0(1, 2))$;
- b.) satisfies the geodesic KMS condition: for every wedge $W = \Lambda W_1$, $\Lambda \in \text{SO}_0(1, 2)$, the partial state

$$\omega_{\upharpoonright \mathcal{A}(W)}(A) \doteq \langle \Omega, A\Omega \rangle , \quad A \in \mathcal{A}(W) ,$$

satisfies the KMS-condition at inverse temperature $\beta = 2\pi r$ with respect to the one-parameter group $t \mapsto \mathbb{U}(\Lambda_W(t/r))$, $t \in \mathbb{R}$.

- iv.) (Additivity). For X a double cone or a wedge, there holds

$$\mathcal{A}(X) = \bigvee_{\mathcal{O} \subset X} \mathcal{A}(\mathcal{O}) . \quad (11)$$

The right hand side denotes the von Neumann algebra generated by the local algebras associated to double cones \mathcal{O} contained in X . (It thus makes sense to define $\mathcal{A}(X)$ for arbitrary regions X by Equ. (11).)

- iv'.) (Weak additivity). For each double cone $\mathcal{O} \subset \mathfrak{dS}$ there holds

$$\bigvee_{\Lambda \in \text{SO}_0(1,2)} \mathcal{A}(\Lambda\mathcal{O}) = \mathcal{A}(\mathfrak{dS}) \quad (= \mathcal{B}(\mathcal{F})) .$$

- v.) (Time-slice axiom [5]). Let G be a causally complete region and let \mathcal{U} be a neighborhood of a geodesic Cauchy surface of G such that $G = \mathcal{U}''$. Then

$$\mathcal{A}(\mathcal{U}) = \mathcal{A}(G) ,$$

where both algebras are defined via Eq. (11). In particular, the algebra of observables located within an arbitrary small time-slice coincides with the algebra of all observables.

Proof. If \mathcal{O}_1 and \mathcal{O}_2 are two space-like separated causally complete, open and bounded regions, then there exists a wedge $W = \Lambda W_1$ such that

$$\mathcal{O}_1 \subset W \quad \text{and} \quad \mathcal{O}_2 \subset W' .$$

Now the interacting net inherits wedge duality

$$\mathcal{A}(W') = \mathcal{A}(W)'$$

from the identity $\mathcal{A}_\circ(W'_1) = \mathcal{A}_\circ(W_1)'$ using (7). These facts imply locality.

Next, let us prove covariance. Let $\Lambda \in SO_0(1, 2)$ be fixed. By construction, the set of all wedges equals $\{\Lambda W_1 \mid \Lambda \in SO_0(1, d)\}$. Thus

$$\begin{aligned} \mathcal{A}(\Lambda \mathcal{O}) &= \bigcap_{\Lambda \mathcal{O} \subset \Lambda W} \mathcal{A}(\Lambda W) = \bigcap_{\mathcal{O} \subset W} \alpha_\Lambda(\mathcal{A}(W)) \\ &= \alpha_\Lambda\left(\bigcap_{\mathcal{O} \subset W} \mathcal{A}(W)\right) = \alpha_\Lambda(\mathcal{A}(\mathcal{O})) , \end{aligned}$$

proving covariance. For the proof of property iii.) we refer the reader to [2].

Next, we will establish property iv.). The inclusion

$$\mathcal{A}(X) \supset \bigvee_{\mathcal{O} \subset X} \mathcal{A}(\mathcal{O}) .$$

is a consequence of isotony. Moreover, if X is a double cone, then X itself is among the double cones on the right hand side, so the inclusion \subset automatically holds. It remains to prove the inclusion \subset if X is a wedge. If $X = W_1$, then $\mathcal{A}(X)$ coincides with $\mathcal{A}_o(X)$, for which the additivity property for the one-particle space implies

$$\mathcal{A}_o(X) = \bigvee_{\substack{R \in SO(2) \\ RI \subset I_+}} \mathcal{A}_o(R\mathcal{O}_I) ,$$

where I is any (arbitrarily small) interval contained in $I_+ \doteq W_1 \cap S^1$, and $\mathcal{O}_I = I''$. Thus,

$$\mathcal{A}(X) = \bigvee_{\substack{R \in SO(2) \\ RI \subset I_+}} \mathcal{A}(R\mathcal{O}_I) \subset \bigvee_{\mathcal{O} \subset X} \mathcal{A}(\mathcal{O}) .$$

Thus, for $X = W_1$ the inclusion \subset holds. By covariance, it also holds if X is any other wedge.

Property iv'.) follows from a similar argument: for each double cone $\mathcal{O} \subset \mathbb{dS}$ there exists a Lorentz transformation $\Lambda_0 \in SO_0(1, 2)$ such that $\mathcal{O} = \Lambda_0 \mathcal{O}_I$ for some open interval $I \subset S^1$. Now

$$\begin{aligned} \bigvee_{\Lambda \in SO_0(2,1)} \mathcal{A}(\Lambda \mathcal{O}) &= \bigvee_{\Lambda \in SO_0(2,1)} \mathcal{A}(\Lambda \Lambda_0 \mathcal{O}_I) \\ &= \bigvee_{\Lambda \in SO_0(2,1)} \mathcal{A}(\Lambda \mathcal{O}_I) \supset \bigvee_{R \in SO(2)} \mathcal{A}(R\mathcal{O}_I) \\ &= \bigvee_{R \in SO(2)} \mathcal{A}_o(R\mathcal{O}_I) = \mathcal{B}(\mathcal{F}) . \end{aligned}$$

Again, the last equality relies on the additivity property for the one-particle space. Hence, property iv'.) is established.

Let us prove property v.), assuming in a first step that the Cauchy surface is an interval I contained in the “equator” S^1 , and U is a neighbourhood of I in G . If I is smaller than a half circle, then G is a double cone and $\mathcal{A}(G) = \mathcal{A}_o(G)$. Pick any double cone $\mathcal{O} \subset U$ with base on S^1 . Then the additivity property of the free net implies that

$$\mathcal{A}_o(G) = \bigvee_{\substack{R \in SO(2) \\ R\mathcal{O} \subset U}} \mathcal{A}_o(R\mathcal{O}).$$

Now $\mathcal{A}_o(R\mathcal{O})$ coincides with $\mathcal{A}(R\mathcal{O})$, and hence the above identity implies $\mathcal{A}(G) \subset \mathcal{A}(U)$. The other inclusion follows from isotony.

If the interval I is a half circle on S^1 or larger, then $\mathcal{A}(G)$ is generated by all local algebras associated with some $\mathcal{O} \subset G$. Each such \mathcal{O} is contained in the double cone with base $I(\mathcal{O}) \doteq \Gamma(\mathcal{O}) \cap S^1$ in the equator S^1 . By the previous step,

$$\mathcal{A}(\mathcal{O}_{I(\mathcal{O})}) = \mathcal{A}(U_{I(\mathcal{O})}),$$

where $U_I \doteq U \cap \mathcal{O}_I$. Thus,

$$\mathcal{A}(G) = \bigvee_{\mathcal{O} \subset G} \mathcal{A}(\mathcal{O}) \subset \bigvee_{\mathcal{O} \subset G} \mathcal{A}(U_{I(\mathcal{O})}) \subset \mathcal{A}(U).$$

Again, the other inclusion is trivial.

Now the geodesic Cauchy surfaces are precisely the Lorentz transforms of intervals on the equator S^1 , see [6]. Thus, the general case follows by covariance. \square

Remarks 5.1.

- i.) We note that the properties i.) – iii.) are valid, even without property (9).
- ii.) It has been shown by Borchers and Buchholz that if one assumes the geodesic KMS condition to hold for some $\beta > 0$, then automatically $\beta = 2\pi r$; see [3, Theorem 6.2].

The so-called Reeh-Schlieder property, namely the statement that for any open region $\mathcal{O} \subset dS$ there holds

$$\overline{\mathcal{A}(\mathcal{O})\Omega} = \mathcal{F},$$

follows directly from these axioms [3].

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